

RANDOMNESS VIA INFINITE COMPUTATION AND EFFECTIVE DESCRIPTIVE SET THEORY

MERLIN CARL AND PHILIPP SCHLICHT

ABSTRACT. We study randomness beyond Π_1^1 -randomness and its Martin-Löf type variant, introduced in [HN07] and further studied in [BGM].

The class given by the infinite time Turing machines (ITTMs), introduced by Hamkins and Kidder, is strictly between Π_1^1 and Σ_2^1 . We prove that the natural randomness notions associated to this class have several desirable properties resembling those of the classical random notions such as Martin-Löf randomness, and randomness notions defined via effective descriptive set theory such as Π_1^1 -randomness. For instance, mutual randoms do not share information and can be characterized as in van Lambalgen's theorem. We also obtain some differences to the hyperarithmetic setting. Already at the level of Σ_2^1 , some properties of randomness notions are independent [CS17].

Towards the results about randomness, we prove the following analogue to a theorem of Sacks. If a real is infinite time Turing computable relative to all reals in some given set of reals with positive Lebesgue measure, then it is already infinite time Turing computable. As a technical tool, we prove facts of independent interest about random forcing over admissible sets and increasing unions of admissible sets. These results are also useful for more efficient proofs of some classical results about hyperarithmetic sets.

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1. INTRODUCTION

Algorithmic randomness studies formal notions that express the intuitive concept of an *arbitrary* or *random* infinite bit sequence with respect to Turing programs. The most prominent such notion is *Martin-Löf randomness* (ML). A real number, i.e. a sequence of length the natural numbers with values 0 and 1, is ML-random if and only if it is not contained in a set of Lebesgue measure 0 that can be effectively approximated by a Turing

machine in a precise sense. We refer the reader to comprehensive treatments of this topic in [DH10, Nie09].

Martin-Löf already suggested that the classical notions of randomness are too weak. Moreover, Turing computability is relatively weak in comparison with notions in descriptive set theory. Therefore higher notions of randomness have been considered, for instance, computably enumerable sets are replaced with Π_1^1 sets (see [HN07, BGM]). These notions were recently studied in [BGM], and in particular the authors defined a continuous relativization which allowed them to prove a variant of van Lambalgen's theorem for Π_1^1 -ML-random. We will use this or the Martin-Löf variant of ITTM-random reals in Section 4.3.

There are various desirable properties for a notion of randomness, which many of the formal notions possess, and which can serve as criteria for the evaluation of such a notion. For instance, different approaches to the notion of randomness, such as not having effective rare properties, being incompressible or being unpredictable are often equivalent. Van Lambalgen's theorem states that each half of a random sequence is random with respect to the other half. Moreover, there is often a universal test. For instance ML-randomness and its Π_1^1 -variant (see [HN07] and [BGM] for the relativization) satisfy these conditions. Some types of random reals are not informative and real numbers that are mutually random do not share any nontrivial information. This does not hold for ML-randomness and its variant at the level of Π_1^1 , but it does hold for Π_1^1 -randomness and the notion of ITTM-randomness studied in this paper.

Higher randomness studies properties of classical randomness notions for higher variants. Various results can be extended to higher randomness notions, assuming sufficiently large cardinals (see e.g. [CY]). However, already at the level of Σ_2^1 , many properties of randomness notions are independent [CS17]. Therefore we consider classes strictly between Π_1^1 and Σ_2^1 .

The infinite time Turing machines introduced by Hamkins and Kidder (see [HL00]) combine the appeal of machine models with considerable strength. The notions decidable, semi-decidable, computable, writable etc. will refer to these machines. The strength of these machines is strictly above Π_1^1 and therefore, this motivates the consideration of notions of randomness based on ITTMs. This project was started in [CS17] and continued in [Carb, Cara].

We consider the following notions of randomness for a real.

- ITTM-random: avoids every semidecidable null set,
- ITTM-decidable random: avoids every decidable null set,
- ITTM_{ML}-random: like ML-randomness, but via ITTMs instead of Turing machines.

With respect to the above criteria, they perform differently. As we show below, all notions satisfy van Lambalgen's theorem. We will see that there is a universal test for ITTM-randomness and ITTM_{ML}-randomness, but not for ITTM-decidable randomness, and we will relate these notions to randomness over initial segments of the constructible hierarchy. A new phenomenon for ITTMs compared to the hyperarithmetic setting is the existence of *lost melodies*, i.e. non-computable recognizable sets (see [HL00]). We will see that lost melodies are not computable from any ITTM-random real. Moreover, we observe that as in [HN07], ITTM_{ML}-randomness is equivalent to a notion of incompressibility of the finite initial segments of the string.

The first main result is an analogue to a result of Sacks [DH10, Corollary 11.7.2]: computability relative to all elements of a set of positive Lebesgue measure implies computability (asked in [CS17, Section 3]). This result is used in several proofs below.

Theorem 1.1. (Theorem 3.16) Suppose that A is a subset of the Cantor space ${}^\omega 2$ with $\mu(A) > 0$ and a real x is ITTM-computable from all elements of A . Then x is ITTM-computable.

The proof rests on phenomena for infinite time computations that have no analogue in the context of Turing computability, in particular the difference between writable, eventually writable and accidentally writable reals (see Definition 3.1 or [Wel09]).

We state some other main results. We obtain a variant for the stronger *hypermachines* with Σ_n -limit rules [?] in Theorem 3.18. We prove a variant of the previous theorem for recognizable sets.¹ Thus we answer several questions posed in [Car15, Section 5] and [Carb, Section 6].

Theorem 1.2. (Theorem 3.20) Suppose that A is a subset of the Cantor space ${}^\omega 2$ with $\mu(A) > 0$ and a real x is ITTM-recognizable from all elements of A . Then x is ITTM-recognizable.

The next result characterizes ITTM-randomness by the values of an ordinal Σ that is associated to ITTM-computations, the supremum of the ordinals coded by accidentally writable reals, i.e. reals that can be written on the tape at some time in some computation.

Theorem 1.3. (Theorem 4.4) A real x is ITTM-random if and only if it is random over L_Σ and $\Sigma^x = \Sigma$.

The following is a desirable property of randomness that holds for Π_1^1 -randomness, but not for Martin-Löf randomness. The property states that mutual randoms do not share non-computable information. Here, two reals are considered random if their join is random.

Theorem 1.4. (Theorem 4.5) If x is computable from both y and z , and y and z are mutual ITTM-randoms, then x is computable.

We further analyze a decidable variant of ITTM-random that is analogous to Δ_1^1 -random. We characterize this notion in Theorem 4.7 and prove an analogue to Theorem 4.5 and to van Lambalgen's theorem for this variant.

All results in this paper, except for the Martin-Löf variant in Section 4.3, work for Cohen reals instead of random reals, often with much simpler proofs, which we do not state explicitly.

The main tool is a variant of random forcing suitable for models of weak set theories such as Kripke-Platek set theory. Previously, some results were formulated for the ideal of meager sets instead of the ideal of measure null sets, since the proofs use Cohen forcing and this is a set forcing in such models. Random forcing, on the other hand, is a class forcing in this contexts, and it is worthwhile to note that random generic is different from random over these models (see [Yu11]). These difficulties are overcome through an alternative definition of the forcing relation, which we call the *quasi-forcing relation*.

As a by-product, the analysis of random forcing allows more efficient proofs of classical results of higher recursion theory, such as Sacks' theorem that $\{x \mid \omega_1^x > \omega_1^{\text{ck}}\}$ is a null set, although the quasi-generics used here are quite different from generics used in forcing (see [Yu11, Remark after Theorem 6.6]).

We assume some familiarity with infinite time Turing machines (see [HL00]), randomness (see [Nie09]) and admissible sets (see [Bar75]). In Section 4.3 we will refer to some proofs in [HN07, Section 3] and [BGM, Section 3]. Moreover, we frequently use the Gandy-Spector theorem to represent Π_1^1 sets (see [Hjo10, Theorem 5.5]).

The paper is structured as follows. In Section 2, we discuss random forcing over admissible sets and limits of admissible sets. In Section 3, we prove results about infinite

¹An element x of ${}^\omega 2$ is ITTM-recognizable if $\{x\}$ is ITTM-decidable (see Definition 3.19).

time Turing machines and computations from non-null sets. This includes the main theorem. In Section 4, we use the previous results to prove desirable properties of randomness notions.

We would like to thank Laurent Bienvenu for allowing us to include joint results with the first author on ITTM-genericity in Section 4.2. Moreover, we would like to thank Andre Nies, Philip Welch and Liang Yu for discussions related to the topic of this paper.

2. RANDOM FORCING OVER ADMISSIBLE SETS

In this section, we present some results about random forcing over admissible sets and unions of admissible sets that are of independent interest. This is essential for the following proofs. The results simplify the approach to forcing over admissible sets (see [Sac90]) by avoiding a ranked forcing language.

We first fix some (mostly standard) notation. A *real* is a set of natural numbers or an element of the Cantor space ${}^\omega 2$. The basic open subsets of the Cantor space ${}^\omega 2$ will be denoted by $U_s = \{x \in {}^\omega 2 \mid s \subseteq x\}$ for $s \in {}^{<\omega} 2$. The Lebesgue measure on ${}^\omega 2$ is the unique Borel measure μ with size $\mu(N_t) = 2^{-|t|}$ for all $t \in {}^{<\omega} 2$. An *admissible set* is a transitive set which satisfies Kripke-Platek set theory with the axiom of infinity. Moreover, an ordinal α is called admissible if L_α is admissible.

2.1. The quasi-forcing relation. We work with the following version of random forcing. If T is a subtree T of ${}^{<\omega} 2$, i.e. a downwards closed subset, let

$$[T] = \{x \in {}^\omega 2 \mid \forall n \ x \restriction n \in T\}$$

denote the set of (cofinal) branches of T . A *perfect subtree* of $2^{<\omega}$ is a subtree without end nodes and cofinally many splitting nodes. We define random forcing as the set of perfect subtrees T of ${}^{<\omega} 2$ with $\mu([T]) > 0$, partially ordered by reverse inclusion. Note that it can be easily shown (but will not be used here), for random forcing in any admissible set, that this partial order is dense in the set of Borel subsets A of ${}^\omega 2$ (given by Borel codes) with $\mu(A) > 0$. Note that random forcing is, in general, a class forcing over admissible sets, and this is the reason why we will need the following results.

Definition 2.1. Suppose that α is an ordinal and $x \in {}^\omega 2$. Then x is *random* over L_α if $x \in A$ for every Borel set A with a Borel code in L_α .

We distinguish between the forcing relation for random forcing over an admissible and the *quasi-forcing relation* defined below. In the definition of the quasi-forcing relation, the condition that a set is dense is replaced with the condition that the union of the conditions has full measure. Hence the quasi-forcing relation corresponds to a random real, i.e. a real which is a member of a class of definable sets of measure 1, for instance all Π_1^1 sets of measure 1. Such reals are sometimes called quasi-generics (see [Ike10]).

This contrasts the notion of random generics in the sense of forcing. The following example shows that these two notions are different. Given any $n \geq 1$, we construct a dense Π_1^1 subset A of the random forcing in $L_{\omega_1^{\text{ck}}}$ with $\mu(\bigcup A) < \frac{1}{n}$. Suppose that a Σ_1 -definable enumeration $\langle B_\alpha \mid \alpha < \omega_1^{\text{ck}} \rangle$ of the Borel codes in $L_{\omega_1^{\text{ck}}}$ for all Borel sets with positive measure and codes in $L_{\omega_1^{\text{ck}}}$ given. We will use the same notation for a set and its code. Moreover, suppose that a partial surjection $f: \omega \rightarrow \omega_1^{\text{ck}}$ is given that is Σ_1 -definable over $L_{\omega_1^{\text{ck}}}$. We define a sequence of Borel sets $A_\alpha \subseteq B_\alpha$ with $0 < \mu(A_\alpha) < 2^{-(i+n+1)}$, where i is least such that $f(i) = \alpha$. Then $A = \bigcup_{\alpha < \omega_1^{\text{ck}}} A_\alpha$ is a Π_1^1 set by the Gandy-Spector theorem [Hjo10, Theorem 5.5]. The difference is illustrated even better by Liang Yu's result [Yu11] that $\omega_1^x > \omega_1^{\text{ck}}$ for any random generic x over $L_{\omega_1^{\text{ck}}}$. Together with a classical result (for a proof, see Lemma 2.13 below) shows that no random generic over $L_{\omega_1^{\text{ck}}}$ avoids every Π_1^1 null set.

Moreover, it is shown below that the quasi-forcing relation for Δ_0 -formulas over admissible sets is definable, while we do not know if this holds for the forcing relation.

We now define Boolean values for the quasi-forcing relation for random forcing. An ∞ -Borel code is a set of ordinals that codes a set built from basic open subsets of ${}^\omega 2$ and their complements by forming intersections and unions of any ordinal length. We will write $\bigvee_{i \in I} x_i$ for the canonical code for the union of the sets coded by x_i for $i \in I$, and similarly for $\bigwedge_{i \in I} x_i$ and $\neg x$.

Definition 2.2. Suppose that L_α is admissible or an increasing union of admissible sets. We define $\llbracket \varphi(\sigma_0, \dots, \sigma_n) \rrbracket$ by induction in L_α , where $\sigma_0, \dots, \sigma_n \in L_\alpha$ are names for random forcing and $\varphi(x_0, \dots, x_n)$ is a formula.

- (a) $\llbracket \sigma \in \tau \rrbracket = \bigvee_{(\nu, p) \in \tau} \llbracket \sigma = \nu \rrbracket \wedge p$.
- (b) $\llbracket \sigma = \tau \rrbracket = (\bigwedge_{(\nu, p) \in \sigma} \llbracket \nu \in \tau \rrbracket \wedge p) \wedge (\bigwedge_{(\nu, p) \in \tau} \llbracket \nu \in \sigma \rrbracket \wedge p)$.
- (c) $\llbracket \exists x \in \sigma_0 \varphi(x, \sigma_0, \dots, \sigma_n) \rrbracket = \bigvee_{(\nu, p) \in \sigma_0} \llbracket \varphi(\nu, \sigma_0, \dots, \sigma_n) \rrbracket \wedge p$.
- (d) $\llbracket \neg \varphi(\sigma_0, \dots, \sigma_n) \rrbracket = \neg \llbracket \varphi(\sigma_0, \dots, \sigma_n) \rrbracket$.
- (e) $\llbracket \exists x \varphi(x, \tau) \rrbracket = \bigcup_{\sigma \in L_\alpha} \llbracket \varphi(\sigma, \tau) \rrbracket$.

We will identify $\llbracket \varphi(\sigma_0, \dots, \sigma_n) \rrbracket$ with the subset of ${}^\omega 2$ that it codes. This quasi-forcing relation is defined as follows.

Definition 2.3. Suppose that α is admissible or a limit of admissibles, p a random condition in L_α , $\varphi(x_0, \dots, x_n)$ formula and $\sigma_0, \dots, \sigma_n$ random names in L_α . We define $p \Vdash^{L_\alpha} \varphi(\sigma_0, \dots, \sigma_n)$ if $\mu([p] \setminus \llbracket \varphi(\sigma_0, \dots, \sigma_n) \rrbracket) = 0$.

Lemma 2.4. Suppose that α is admissible or a limit of admissibles. Then the function which associates a Boolean value to Δ_0 -formulas $\varphi(\sigma_0, \dots, \sigma_n)$ and the forcing relation for random forcing are Δ_1 -definable over L_α .

Proof. The Boolean values are defined by a Δ_1 -recursion and the measure corresponding to a code is definable by a Δ_1 -recursion. This implies that the forcing relation is Δ_1 -definable. \square

Definition 2.5. Suppose that α is an ordinal and $x \in {}^\omega 2$. We define $\sigma^x = \{\nu^x \mid (\nu, p) \in \sigma, x \in [p]\}$ for $\sigma \in L_\alpha$ by induction on the rank.

- (a) The generic extension of L_α by x is defined as $L_\alpha[x] = \{\sigma^x \mid \sigma \in L_\alpha\}$.
- (b) The α th level of the L -hierarchy built over x , with $L_0[x] = \text{tc}(\{x\})$, is denoted by L_α^x .

We will show in Lemma 2.9 and Lemma 2.10 that the sets $L_\alpha[x]$ and L_α^x are equal if x is random over L_α and α is admissible or a limit of admissibles.

Lemma 2.6. Suppose that L_α is admissible or an increasing union of admissible sets, $\sigma_0, \dots, \sigma_n \in L_\alpha$ are names for random forcing, $\varphi(x_0, \dots, x_n)$ is a Δ_0 -formula and x is random over L_α . Then

$$L_\alpha[x] \models \varphi(\sigma_0^x, \dots, \sigma_n^x) \iff x \in \llbracket \varphi(\sigma_0, \dots, \sigma_n) \rrbracket.$$

Moreover, this holds for all formulas if α is countable in L_β and x is random over L_β .

Proof. By induction on the ranks of names and on the formulas. \square

The following is a version of the forcing theorem for the quasi-forcing relation.

Lemma 2.7. Suppose that α is admissible or a limit of admissibles and p is a random condition in L_α . Then $p \Vdash^{L_\alpha} \varphi(\sigma_0, \dots, \sigma_n)$ if and only if $L_\alpha[x] \models \varphi(\sigma_0^x, \dots, \sigma_n^x)$ for all random $x \in [p]$ over L_α .

Proof. Suppose that $p \Vdash^{L_\alpha} \varphi(\sigma_0, \dots, \sigma_n)$ and $x \in [p]$ is random over L_α . Then $\mu([p] \setminus \llbracket \varphi(\sigma_0, \dots, \sigma_n) \rrbracket) = 0$. Since x is random over L_α , $x \in \llbracket \varphi(\sigma_0, \dots, \sigma_n) \rrbracket$. This implies $L_\alpha[x] \models \varphi(\sigma_0^x, \dots, \sigma_n^x)$ by Lemma 2.6.

Suppose that $p \not\Vdash^{L_\alpha} \varphi(\sigma_0, \dots, \sigma_n)$. Then $\mu([p] \setminus \llbracket \varphi(\sigma_0, \dots, \sigma_n) \rrbracket) > 0$. Suppose that $x \in [p] \setminus \llbracket \varphi(\sigma_0, \dots, \sigma_n) \rrbracket$ is random over L_α . Then $L_\alpha[x] \models \neg \varphi(\sigma_0^x, \dots, \sigma_n^x)$ by Lemma 2.6. \square

The following is a version of the truth lemma for the quasi-forcing relation.

Lemma 2.8. Suppose that α is admissible or a limit of admissibles and x is random over L_α . Then $L_\alpha[x] \models \varphi(\sigma^x)$ holds if and only if there is a random condition p in L_α with $x \in [p]$ and $p \Vdash \varphi(\sigma)$.

Proof. Suppose that $x \in [p]$ and $p \Vdash \varphi(\sigma)$. Then $\mu([p] \setminus \llbracket \varphi(\sigma) \rrbracket) = 0$. Since x is random over L_α , $x \in L_\alpha[x] \models \varphi(\sigma^x)$ holds. Then $L_\alpha[x] \models \varphi(\sigma^x)$ by Lemma 2.6.

Suppose that $L_\alpha[x] \models \varphi(\sigma^x)$ holds. Then $x \in \llbracket \varphi(\sigma) \rrbracket$ by Lemma 2.6. Since $\mu(\llbracket \varphi(\sigma) \rrbracket)$ is the supremum of $\mu([p])$, where p is a condition in L_α with $[p] \subseteq \llbracket \varphi(\sigma) \rrbracket$, and x is random over L_α , there is a condition p in L_α with $x \in [p]$. Since $[p] \subseteq \llbracket \varphi(\sigma) \rrbracket$, $p \Vdash^{L_\alpha} \varphi(\sigma)$. \square

2.2. The generic extension. If α is admissible or a limit of admissibles and x is random over L_α , we show that $L_\alpha[x]$ is equal to L_α^x .

Lemma 2.9. Suppose that α is admissible or a limit of admissibles and x is random over L_α . Then for all $\gamma < \alpha$ and all $\sigma \in L_\gamma$, $\sigma^x \in L_{\gamma+2}^x$.

Proof. Suppose that $\sigma \in L_\gamma$ is a name. We define for all $\beta < \gamma$ the β -th approximate evaluation for σ as the function

$$f_{\beta, \sigma} : \text{tc}(\sigma) \cap L_\beta \rightarrow L_\alpha[x]$$

which maps (τ, p) to τ^x if $x \in [p]$ and to \emptyset otherwise. Moreover, let $F_\gamma(\beta) = f_{\beta, \sigma}$ for $\beta < \gamma$.

We will show by simultaneous induction that $f_{\beta, \sigma}, F_\beta \in L_{\beta+2}$ for all $\beta < \gamma$. It will then follow easily that σ^x is definable over $L_{\gamma+1}$ and hence an element of $L_{\gamma+2}$. Suppose that $\beta = \theta + 1$. Then $F_\theta \in L_{\theta+1}$ by the inductive hypothesis. We define $f_{\beta, \sigma}$ over L_θ by

$$f_{\beta, \sigma}(\tau, p) = \{F_\theta(\bar{\tau}) \mid x \in [p], \exists q ((\bar{\tau}, q) \in \tau, x \in [q])\}$$

Then $f_{\beta, \sigma} \in L_\beta = L_{\theta+1}$. Let $F_\beta = F_\theta \cup \{(\theta, f_{\theta, \sigma})\}$. If β is a limit ordinal, we define $f_{\beta, \sigma}$ by

$$f_{\beta, \sigma}((\tau, p)) = \{F_\delta(\bar{\tau}) \mid x \in [p], \delta < \gamma, \tau' \in L_\delta, \exists q((\bar{\tau}, q) \in \tau, x \in [q])\}.$$

To define F_β in the limit case, we proceed as follows. Note that for $\delta < \beta$, F_δ is the unique function which satisfies the following in L_β : $\text{dom}(F) = \delta$, $F(0) = 0$, F is continuous at all limits $\gamma < \delta$, and $F(\eta + 1)$ is defined as in the successor case above for all $\eta < \delta$. It follows that $f_{\gamma, \sigma}$ is definable over $L_{\gamma+1}$ and hence $\sigma^x = f_{\gamma, \sigma}(\sigma) \in L_{\gamma+2}$. \square

Lemma 2.10. Suppose that α is admissible or a limit of admissibles and x is random over L_α . Then $L_\alpha^x \subseteq L_\alpha[x]$.

Proof. It is sufficient to prove this for the case that α is admissible. It is sufficient to show that there is a Σ_1 -definable sequence $\langle \tau_\gamma, \alpha_\gamma \mid \gamma < \alpha \rangle$ such that each τ_γ is a name, α_γ is an ordinal, $\sup_{\gamma < \alpha} \alpha_\gamma = \alpha$, τ_γ is uniformly Σ_1 -definable over L_{α_γ} and $\tau^x = L_\gamma[x]$. Since $L_\gamma[x]$ is transitive, this implies the claim.

Suppose that τ_γ and α_γ are defined. Suppose that $(\sigma, p) \in \tau_\gamma$. Let φ^x denote the relativization of a formula φ to a set x . Since α is admissible, there is a least ordinal $\delta_{\sigma, p}$ such that $\llbracket \varphi^{\tau_\gamma}(\sigma_0, \dots, \sigma_n) \rrbracket \subseteq \delta_{\sigma, p}$ for all formulas $\varphi(x_0, \dots, x_n)$ and all names $\sigma_0, \dots, \sigma_n$ such that there are conditions p_i with $(\sigma_i, p_i) \in \tau_\gamma$ for all $i \leq n$.

Let $\alpha_{\gamma+1} = \sup_{(\sigma,p) \in \tau_\gamma} \delta_{\sigma,p}$. Then $\alpha_{\gamma+1}$ is uniformly Σ_1 -definable from α_γ and τ_γ . Moreover, let $\tau_\gamma^\varphi = \{(\sigma, p) \in \tau_\gamma \mid p \Vdash \phi^{\tau_\gamma}(\sigma)\}$. Then τ_γ^ϕ is uniformly Σ_1 -definable over $L_{\alpha_{\gamma+1}}$. By forming unions at limits, we define the sequence $\langle \tau_\gamma, \alpha_\gamma \mid \gamma < \alpha \rangle$ in a Σ_1 recursion. \square

We now argue that $L_\alpha[x]$ is admissible if α is admissible and x is sufficiently random.

Lemma 2.11. Suppose that α is admissible or a limit of admissibles, and x is random over $L_{\alpha+1}$. Then $L_\alpha[x]$ is admissible or a limit of admissibles, respectively.

Proof. It is sufficient to prove this for the case where α is admissible. Suppose that f is a Σ_1 -definable function over $L_\alpha[x]$ that is cofinal in α and has domain $\eta < \alpha$. We will assume that $\eta = \omega$ to simplify the notation.

Suppose that \dot{x} is a name for the random generic and that \dot{f} is a name for f . Since f is a function in $L_\alpha[x]$ and x is random over $L_{\alpha+1}$,

$$\mu\left(\bigcap_{n \in \omega} [\exists \alpha \ \dot{f}(n) \in L_\alpha[\dot{x}]]\right) > 0,$$

where the Boolean value of existential formulas is defined as a union in the obvious way. Let $\mu(\bigcap_{n \in \omega} [\exists \alpha \ \dot{f}(n) \in L_\alpha[\dot{x}]]) = \epsilon$.

Claim 2.12. $\mu(\bigcap_{n \in \omega} [\exists \alpha \ \dot{f}(n) \in L_\alpha[\dot{x}]] \setminus [\exists g \ \forall n (\dot{f}(n) = g(n))]) = 0$.

Proof. Suppose that $\delta \leq \epsilon$ with $\delta \in \mathbb{Q}$. We consider the Δ_0 -definable function h that maps n to the least $\bar{\alpha} < \alpha$ such that

$$\mu\left(\bigcap_{i \leq n} [\dot{f}(i) \in L_{\bar{\alpha}}[\dot{x}]]\right) \geq \delta$$

and this Σ_1 -statement (i.e. the statement that the measure is at least δ) is witnessed in $L_{\bar{\alpha}}$. Since α is admissible, we obtain some $\gamma < \alpha$ with $\mu(\bigcap_{n \in \omega} [\exists \alpha \ \dot{f}(n) \in L_\gamma]) \geq \delta$ and hence

$$\mu\left(\bigcap_{n \in \omega} [\exists \alpha \ \dot{f}(n) \in L_\alpha] \setminus [\exists g \ \forall n (\dot{f}(n) = g(n))]\right) \leq \epsilon - \delta.$$

\square

Since the set in Claim 2.12 is definable over L_α , this implies the statement of Lemma 2.11 by Lemma 2.6. \square

As an example for how the previous can be applied to prove known theorems, we consider the following classical result (see [Theorem 9.3.9, Nies]). Note that random over $L_{\omega_1^{\text{ck}}}$ in our notation is equivalent to Δ_1^1 -random.

Lemma 2.13. (see [Nie09, Theorem 9.3.9]) A real x is Π_1^1 -random if and only if x is Δ_1^1 -random and $\omega_1^x = \omega_1^{\text{ck}}$.

Proof. We first claim that $\omega_1^x = \omega_1^{\text{ck}}$ for every Π_1^1 -random real. The set of random reals over $L_{\alpha+1}$ has measure 1, and for these reals x , we have $\omega_1^x = \omega_1^{\text{ck}}$ by Lemma 2.11. Moreover $\omega_1^x > \omega_1$ if and only if there is an admissible ordinal in $L_{\omega_1^x}[x]$, hence the set of these reals is Π_1^1 by the Gandy-Spector theorem [Hjo10, Theorem 5.5]. Thus $\omega_1^x = \omega_1^{\text{ck}}$.

In the other direction, let A denote the largest Π_1^1 null set (see [HN07, Theorem 5.2] and Section 4.1 below). Then $A \subseteq \{x \mid \omega_1^x < \omega_1^{\text{ck}}\} \cup \bigcup_{\alpha < \omega_1^{\text{ck}}} A_\alpha$, where A_α is a Borel set with a code in $L_{\omega_1^{\text{ck}}}$, by the Gandy-Spector theorem [Hjo10, Theorem 5.5]. Since A is the largest Π_1^1 null set, equality holds. If x is Δ_1^1 -random and $\omega_1^x = \omega_1^{\text{ck}}$, then $x \notin A_\alpha$ for all $\alpha < \omega_1^{\text{ck}}$ and hence $x \notin A$. \square

2.3. Side-by-side randoms. Two reals x, y are *side-by-side random* over L_α if $\langle x, y \rangle$ is random over L_α for the Lebesgue measure on ${}^\omega 2 \times {}^\omega 2$. The following Lemma 2.16 is analogous to known results for arbitrary forcings over models of set theory, however the classical proof does not work in our setting.

Lemma 2.14. Suppose that x, y are side-by-side random over L_α . Then x is random over L_α .

Proof. Suppose that A is a Borel subset of ${}^\omega 2$ with Borel code in L_α . Then $\langle x, y \rangle \in A \times {}^\omega 2$. Hence $x \in A$. \square

Lemma 2.15. Suppose that $\langle A_s \mid s \in {}^{<\omega} 2 \rangle$ is a sequence of Lebesgue measurable subsets of ${}^\omega 2$ such that $A_t \subseteq A_s$ for all $s \subseteq t$ in ${}^{<\omega} 2$ and $\mu(\bigcap_n A_{x \upharpoonright n}) = 0$ for all $x \in {}^\omega 2$. Then for every $\epsilon > 0$, there is some n such that for all $s \in {}^n 2$, $\mu(A_s) < \epsilon$.

Proof. If the claim fails, then the tree $T = \{s \in {}^{<\omega} 2 \mid \mu(A_s) \geq \epsilon\}$ is infinite. By König's lemma, T has an infinite branch $x \in {}^\omega 2$. Then $\mu(\bigcap_n A_{x \upharpoonright n}) \geq \epsilon$, contradicting the assumption. \square

We use the forcing theorem for random forcing over admissible sets L_α to prove an analogue to the fact that the intersection of mutually generic extensions is equal to the ground model.

Lemma 2.16. Suppose that L_α is admissible or an increasing union of admissible sets and that x, y are side-by-side random over L_α . Then $L_\alpha[x] \cap L_\alpha[y] = L_\alpha$.

Proof. Let \mathbb{P} denote the random forcing on ${}^\omega 2$ in L_α and \mathbb{Q} the random forcing on ${}^\omega 2 \times {}^\omega 2$ in L_α . Suppose that $z \in L_\alpha[x] \cap L_\alpha[y]$. Moreover, suppose that \dot{x}, \dot{y} are \mathbb{P} -names for z with $\dot{x}^x = z$ and $\dot{y}^y = z$. We can assume that \dot{x}, \dot{y} are \mathbb{Q} -names by identifying them with the \mathbb{Q} -names induced by \dot{x}, \dot{y} . Then every Borel subset of ${}^\omega 2$ that occurs in \dot{x} is of the form $A \times {}^\omega 2$ and every Borel subset of ${}^\omega 2$ occurring in \dot{y} is of the form ${}^\omega 2 \times A$.

Claim 2.17. No condition p forces over L_α that $\dot{x} = \dot{y}$.

Proof. Suppose that $p \Vdash \dot{x} = \dot{y}$ and $\mu([p]) \geq \epsilon > 0$. Then $p \Vdash \bigvee_{s \in {}^{k_2} 2} \dot{x} \upharpoonright k = \dot{y} \upharpoonright k = s$ by Lemma 2.7. Let $A_s = \llbracket \dot{x} \upharpoonright k = s \rrbracket$ and $B_s = \llbracket \dot{y} \upharpoonright k = s \rrbracket$. Then $\mu([p] \setminus \bigcup_{s \in {}^{n_2} 2} (A_s \times B_s)) = 0$ by Lemma 2.6.

There is some n such that $\mu(A_s) < \epsilon$ for all $s \in {}^n 2$ by Lemma 2.15. Since $\sum_{s \in {}^{n_2} 2} \mu(B_s) = 1$, $\sum_{s \in {}^{n_2} 2} \mu(A_s) \mu(B_s) < \epsilon$. The assumption $p \Vdash \dot{x} = \dot{y}$ implies that $\mu([p] \setminus \bigcup_{s \in {}^{n_2} 2} A_s \times B_s) = 0$. Hence $\mu([p]) \leq \mu(\sum_{s \in {}^{n_2} 2} \mu(A_s) \mu(B_s)) < \epsilon$, contradicting the assumption that $\mu([p]) \geq \epsilon$. \square

This completes the proof of Lemma 2.16. \square

3. COMPUTATIONS FROM NON-NULL SETS

In this section, we prove an analogue to the following result of Sacks: any real that is computable from all elements of a set of positive measure is itself computable. This is essential to analyze randomness notions later.

3.1. Facts about infinite time Turing machines. An infinite time Turing machine (ITTM) is a Turing machine that is allowed to run for an arbitrary ordinal time, with the rule of forming the inferior limit in each tape cell and of the (numbered) states in each limit step of the computation. The inputs and outputs of such machines are reals.

We recall some basic facts about these machines (see [HL00, Wel09]). The computable sequences are here called *writable* to distinguish this from the following concepts of computability. These notions from [HL00] are interesting on their own and will be essential in the following proofs via results in [Wel09].

Definition 3.1. (See [HL00])

- (a) A real x is *writable* (or *computable*) if and only if there is an ITTM-program P such that P , when run on the empty input, halts with x written on the output tape.
- (b) A real x is *eventually writable* if and only if there is an ITTM-program P such that P , when run on the empty input, has from some point of time on x written on the output tape and never changes the content of the output tape from this time on.
- (c) A real x is *accidentally writable* if and only if there is an ITTM-program P such that P , when run with empty input, has x written on the output tape at some time (but may overwrite this later on).

We write $P^x \downarrow = i$ if P^x halts with output i . The notation Σ_n will always refer to the standard Levy hierarchy, obtained by counting the number of quantifier changes around a Δ_0 kernel.

The ordinal λ is defined as the supremum of the halting times of ITTM-computations (i.e. the *clockable ordinals*), and equivalently [Wel00, Theorem 1.1] the supremum of the writable ordinals, i.e. the ordinals coded by writable reals. Moreover, ζ is defined as the supremum of the eventually writable ordinals, and Σ is the supremum of the accidentally writable ordinals. The ordinals λ^x , ζ^x and Σ^x are defined relative to an oracle x .

We will use the following theorem by Welch [Wel09, Theorem 1, Corollary 2].

Theorem 3.2. (see [Wel09, Theorem 1, Corollary 2]) Suppose that y is a real. Then $\lambda^y, \zeta^y, \Sigma^y$ have the following properties.

- (1) $L_{\zeta^y}[y]$ is the set of writable reals in y .
- (2) $L_{\zeta^y}[y]$ is the set of eventually writable reals in y .
- (3) $L_{\Sigma^y}[y]$ is the set of accidentally reals in y .

Moreover $(\lambda^y, \zeta^y, \Sigma^y)$ is the lexically minimal triple of ordinals with

$$L_{\lambda^y}[y] \prec_{\Sigma_1} L_{\zeta^y}[y] \prec_{\Sigma_2} L_{\Sigma^y}[y].$$

It is worthwhile to note that the precise definition of the Levy hierarchy is important for the reflection in Theorem 3.2. The characterization of λ , ζ and Σ fails if we allow arbitrary additional bounded quantifiers in the Levy hierarchy, since this variant of Σ_2 -formulas allows to express the fact that a set is admissible. However, L_ζ is admissible [Wel09, Fact 2.2], but L_Σ is not admissible [Wel09, Lemma 6].

We will also use the following information about λ , ζ and Σ .

Theorem 3.3. (a) If the output of an ITTM-program P stabilizes, then it stabilizes before time ζ .

- (b) All non-halting ITTM-computations loop from time Σ on.
- (c) λ and ζ are admissible limits of admissible ordinals (and more).
- (d) In L_λ every set is countable, and the same holds for L_ζ and L_Σ .

Moreover, all of these statements relativize to oracles.

The proofs can be found in [HL00, Wel09]. We will write $x \leq_w y$, $x \leq_{ew} y$, $y \leq_{aw} y$ to indicate that x is writable, eventually writable or accidentally writable, respectively, in the oracle y .

Lemma 3.4. The following are equivalent for a subset A of ${}^\omega 2$.

- (a) A is ITTM-semidecidable.
- (b) There is Σ_1 -formula $\varphi(x)$ such that for all $x \in {}^\omega 2$, $x \in A$ if and only if $L_{\lambda^x}[x] \models \varphi(x)$.

Proof. In the forward direction, the Σ_1 -formula simply states the existence of a halting computation. In the other direction, we can search for a writable code for an initials segment of $L_{\lambda^x}[x]$ which satisfies $\varphi(x)$, using the fact that every set in $L_{\lambda^x}[x]$ has a writable code in x by Theorem 3.2. \square

We call a subset of $2^{<\omega}$ *enumerable* if there is an ITTM listing its elements. It follows from Lemma 3.4 that it is equivalent for a subset A of $2^{<\omega}$ that A is semidecidable, A is enumerable or that A is Σ_1 -definable over L_λ .

Note that every ITTM-semidecidable set is absolutely Δ_2^1 , i.e. it remains Δ_2^1 with the same definition in any inner model and in any forcing extension. Therefore such sets are Lebesgue measurable and have the property of Baire by [Kan09, Exercise 14.4].

3.2. Preserving reflection properties by random forcing. The following reflection argument is an essential step in the proof of the preservation of λ , ζ and Σ (see Section 3.3 below) with respect to random forcing. We show that for admissibles or limits of admissibles $\alpha < \beta$, the statements $L_\alpha \prec_{\Sigma_1} L_\beta$ and $L_\alpha \prec_{\Sigma_2} L_\beta$ are preserved to generic extensions by sufficiently random reals (i.e. $L_\alpha[x] \prec_{\Sigma_n} L_\beta[x]$ holds for all sufficiently random reals x).

Definition 3.5. Suppose that A is a Lebesgue measurable subset of ${}^\omega 2$. An element x of ${}^\omega 2$ is a (*Lebesgue*) *density point* of A if $\lim_n \frac{\mu(A \cap U_{x \upharpoonright n})}{\mu(U_{x \upharpoonright n})} = 1$. Let $D(A)$ denote the set of density points of A .

We will often use the following version of Lebesgue's density theorem.

Theorem 3.6. (Lebesgue, see [AC13, Section 8]) If A is any Lebesgue measurable subset of ${}^\omega 2$, then $\mu(A \triangle D(A)) = 0$.

We now prove Σ_1 reflection and then Σ_2 -reflection in random extensions, from a stronger hypothesis.

Lemma 3.7. Suppose that $\alpha < \beta$, β is admissible or a limit of admissibles and $L_\alpha \prec_{\Sigma_1} L_\beta$. If x is random over L_β , then $L_\alpha[x] \prec_{\Sigma_1} L_\beta[x]$.

Proof. Note that the assumption $L_\alpha \prec_{\Sigma_1} L_\beta$ implies that L_α is admissible. To see this, note that for any Σ_1 -definable function $f: z \rightarrow L_\alpha$ over L_α , the set L_α is a witness for the Σ_1 -collection scheme for f in L_β . It follows from the assumption $L_\alpha \prec_{\Sigma_1} L_\beta$ that there is a set in L_α witnessing the Σ_1 -collection scheme for f in L_α , and in particular $f \in L_\alpha$.

Suppose that $L_\beta[x] \models \exists v \varphi(v, \tau^x)$, where $\varphi(v, w)$ is a Δ_0 -formula and $\tau \in L_\alpha$. We choose a witness $y \in L_\beta$ such that $\varphi(y, \tau^x)$ holds in $L_\beta[x]$. Moreover, suppose that $\sigma \in L_\beta$ is a name with $\sigma^x = y$. Let $A = \llbracket \varphi(\sigma, \tau) \rrbracket$ and let A_n denote the set of $s \in {}^{<\omega} 2$ with $\frac{\mu(A \cap U_s)}{\mu(U_s)} > 1 - 2^{-n}$, for $n \in \omega$.

In the next claim, we conclude from the Lebesgue density theorem 3.6 that A is almost everywhere covered by the sets U_s for $s \in A_n$. By an *antichain* in $2^{<\omega}$ we mean a subset of $2^{<\omega}$ whose elements are pairwise incomparable with respect to \subseteq . Moreover, an *antichain* in a subset C of $2^{<\omega}$ is an antichain $\bar{C} \subseteq C$. A *maximal antichain* in C is maximal with respect to \subseteq among all antichains in C .

Claim 3.8. If A^* is a maximal antichain in A_n , then $\mu(A \cap \bigcup_{s \in A^*} U_s) = \mu(A)$.

Proof. Suppose that the claim fails and hence $\mu(A \setminus \bigcup_{s \in A^*} U_s) > 0$. Then there is a density point z of $A \setminus \bigcup_{s \in A^*} U_s$ by the Lebesgue density theorem 3.6. Hence there is some k with $\frac{\mu(A \cap U_{z \upharpoonright k})}{\mu(U_{z \upharpoonright k})} > 1 - 2^{-n}$ and thus $t := z \upharpoonright k \in A_n$, by the definition of A_n . However, t is incomparable with all elements of A^* , since $z \notin \bigcup_{s \in A^*} U_s$. This contradicts the assumption that A^* is maximal. \square

We choose a maximal antichain A_n^* in A_n for each n . Since A has a Borel code in L_β , we can choose A_n^* such that the sequence $\langle A_n^* \mid n \in \omega \rangle$ is an element of L_β .

We now aim to reflect the Σ_1 -statement $\exists v \varphi(v, \tau)$ from $L_\beta[x]$ to $L_\alpha[x]$. Note that we do not have σ and A available in L_α , but will instead obtain a name in L_α from σ by reflection (i.e. by using the assumption that $L_\alpha \prec_{\Sigma_1} L_\beta$). The following argument ensures

that there is in fact a subset B of A in L_β with full measure relative to A which witnesses the reflection, i.e. for randoms in B over L_β , the statement reflects.

Suppose that $s \in A_n$ is given. We consider the Σ_1 -formula $\psi_n(s)$ which states that there is a condition p such that $[p] \subseteq U_s$, $\frac{\mu([p] \cap U_s)}{\mu(U_s)} > 1 - 2^{-n}$ and $\exists \nu (p \Vdash \varphi(\nu, \tau))$. Since $s \in A_n$, $\psi_n(s)$ holds in L_β , and therefore in L_α , by the assumption $L_\alpha \prec_{\Sigma_1} L_\beta$.

Let p_s^n denote the $<_L$ -least condition in L_α witnessing $\psi_n(s)$ (in fact any choice would work, as long as the sequence $\langle p_s^n \mid n \in \omega \rangle$ is an element of L_β). Let $B_n = \bigcup_{s \in A_{2n}^*} [p_s^{2n}]$ and $B = \bigcup_{n \in \omega} B_n$.

Claim 3.9. $\mu(A \setminus B) = 0$.

Proof. We have $\frac{\mu([p_s^{2n}] \cap U_s)}{\mu(U_s)} > 1 - 2^{-2n}$ for all $s \in A_{2n}$ by the choice of p_s^{2n} , and $\frac{\mu(A \cap U_s)}{\mu(U_s)} > 1 - 2^{-2n}$ for all $s \in A_{2n}$ by the definition of A_{2n} . Hence $\frac{\mu(A \cap B_n \cap U_s)}{\mu(U_s)} > 1 - 2^{-n}$ for all $s \in A_{2n}$, by the definition of B_n . Therefore

$$\frac{\mu(A \cap B_n \cap U_s)}{\mu(A \cap U_s)} \geq \frac{\mu(A \cap B_n \cap U_s)}{\mu(U_s)} > 1 - 2^{-n}.$$

Moreover

$$\mu\left(\bigcup_{s \in A_n^*} (A \cap U_s)\right) = \mu\left(A \cap \bigcup_{s \in A_n^*} U_s\right) = \mu(A)$$

by Claim 3.8. Hence the sets U_s for $s \in A_{2n}^*$ partition A up to a null set. By applying the previous inequality separately for each $s \in A_{2n}^*$, we obtain $\frac{\mu(A \cap B_n)}{\mu(A)} > 1 - 2^{-n}$. Hence $\frac{\mu(A \cap B)}{\mu(A)} = 1$ and $\mu(A \setminus B) = 0$. \square

Since A has a Borel code in L_β and therefore $\langle A_n \mid n \in \omega \rangle$ is an element of L_β , there is a sequence $\langle b_n \mid n \in \omega \rangle \in L_\beta$ such that b_n is a Borel code for B_n . Therefore $B = \bigcup_{n \in \omega} B_n$ has a Borel code in L_β .

Claim 3.10. $L_\alpha[x] \models \exists v \varphi(v, \tau^x)$.

Proof. Recall that $\varphi(y, \tau^x)$ holds in $L_\beta[x]$ and $A = \llbracket \varphi(\sigma, \tau) \rrbracket$, therefore $x \in A$ by Lemma 2.6. Since x is random over L_β by the assumption, and we have already proved that $\mu(A \triangle B) = 0$, we have $x \in B$. Then there is some n with $x \in B_n$. By the definition of B_n , there is some $s \in A_{2n}$ with $x \in [p_s^{2n}] \cap A$. By the definition of p_s^{2n} , there is a name $\nu \in L_\alpha$ such that $p_s^{2n} \Vdash^{L_\alpha} \varphi(\nu, \tau)$. Since $x \in [p_s^{2n}]$, Lemma 2.7 implies that $L_\alpha[x] \models \varphi(\nu^x, \tau)$. \square

Hence the statement $\exists v \varphi(v, \tau^x)$ reflects to $L_\alpha[x]$. \square

We now move to the preservation of Σ_n -reflection under an appropriate hypothesis. The next result shows that the statement $L_\alpha \prec_{\Sigma_n} L_\beta$ is preserved for sufficiently random reals x , i.e. $L_\alpha[x] \prec_{\Sigma_n} L_\beta[x]$ holds in the generic extension. We first need the following lemma.

Lemma 3.11. Suppose that α is admissible or a limit of admissible ordinals, $t \in {}^{<\omega}2$, $\sigma \in L_\alpha$, $\epsilon \in \mathbb{Q}$, $n \geq 1$ and φ is a formula. The formulas in the following claims have the parameters t, σ and ϵ . Let $m_{\sigma,t} = \mu(\llbracket \varphi(\sigma) \rrbracket \cap U_t)$.

- (1) If φ is Σ_n , then
 - (a) $m_{\sigma,t} > \epsilon$ is equivalent to a Σ_n -formula.
 - (b) $m_{\sigma,t} \leq \epsilon$ is equivalent to a Π_n -formula.
- (2) If φ is Π_n , then
 - (a) $m_{\sigma,t} < \epsilon$ is equivalent to a Π_n -formula.
 - (b) $m_{\sigma,t} \geq \epsilon$ is equivalent to a Σ_n -formula.

Proof. For Δ_0 -formulas φ , the claim holds since the function mapping σ to $\llbracket \varphi(\sigma) \rrbracket$ is Δ_1 -definable in σ .

Suppose that $\varphi(x, y)$ is a Π_n -formula. We aim to prove the first claim for the formula $\exists x \varphi(x, y)$.

We have $\mu(\llbracket \exists x \varphi(x, y) \rrbracket \cap U_t) > \epsilon$ if and only if there is some k and some $\sigma_0, \dots, \sigma_k$ such that $\mu(\llbracket \bigvee_{i \leq k} \varphi(\sigma_i, \tau) \rrbracket \cap U_t) > \epsilon$. By the Lebesgue density theorem 3.6, the last inequality is equivalent to the statement that there is some l , a sequence t_0, \dots, t_l of pairwise incompatible extensions of t and some $\epsilon_0, \dots, \epsilon_l \in \mathbb{Q}$ such that $\epsilon = \sum_{i \leq k} \epsilon_i$ and for all $j \leq l$, there is some $i \leq k$ such that $\mu(\llbracket \varphi(\sigma_i, y) \rrbracket \cap U_{t_j}) > \epsilon_i$. Using a universal Σ_n -formula, we obtain an equivalent Σ_n -statement.

We have $\mu(\llbracket \exists x \varphi(x, y) \rrbracket \cap U_t) \leq \epsilon$ if and only if for all $\sigma_0, \dots, \sigma_k$, $\mu(\llbracket \bigvee_{i \leq k} \varphi(\sigma_i, \tau) \rrbracket) \leq \epsilon$. This is a Π_n -statement by argument in the previous case.

The second claim follows by switching to negations. \square

Lemma 3.12. Suppose that $\alpha < \beta$, β is admissible or a limit of admissibles, $n \geq 1$ and $L_\alpha \prec_{\Sigma_n} L_\beta$. Suppose that β is countable in L_γ and that x is random over L_γ . Then $L_\alpha[x] \prec_{\Sigma_n} L_\beta[x]$.

Proof. Note that the assumption $L_\alpha \prec_{\Sigma_1} L_\beta$ implies that L_α is admissible, as in the proof of Lemma 3.7.

Suppose that the statement $\exists u \varphi(u, \tau^x)$ holds in $L_\beta[x]$, where $n = m + 1$, φ is Π_m and $\tau \in L_\alpha$. Suppose that σ_0 is a name in L_β with $L_\beta[x] \models \varphi(\sigma_0^x, \tau^x)$.

Let $A = \llbracket \varphi(\sigma_0, \tau) \rrbracket$. Since β is countable in L_γ , A has a Borel code in L_γ . It follows from Lemma 2.7 that $x \in A$ and $\mu(A) > 0$. Let A_n denote the set of $s \in {}^{<\omega}2$ such that

$$\frac{\mu(A \cap U_s)}{\mu(U_s)} > 1 - 2^{-n}.$$

Claim 3.13. Suppose that A^\star is a maximal antichain in A_n . Then $\mu(A \cap \bigcup_{s \in A^\star} U_s) = \mu(A)$.

Proof. The proof is identical to the proof of Claim 3.8 via the Lebesgue density theorem 3.6. \square

We choose a maximal antichain A_n^\star in A_n for each n . Since A has a Borel code in $L_{\beta+1} \subseteq L_\gamma$, it is possible to choose A_n^\star such that the sequence $\langle A_n^\star \mid n \in \omega \rangle$ is an element of L_γ .

Let $B_\sigma = \llbracket \varphi(\sigma, \tau) \rrbracket$. Then $A = B_{\sigma_0}$. We consider the statement $\psi_n(s)$ stating that there is some name σ such that $\frac{\mu(B_\sigma \cap U_s)}{\mu(U_s)} > 1 - 2^{-n}$. This is a Σ_n -statement by Lemma 3.11.

Since $s \in A_n$, $\psi_n(s)$ holds in L_β . Since $L_\alpha \prec_{\Sigma_n} L_\beta$, this implies that $\psi_n(s)$ holds in L_α . Let σ_s^n denote the $<_L$ -least name in L_α witnessing $\psi_n(s)$, for $s \in A_n$ (in fact any choice would work, as long as the sequence $\langle \sigma_s^n \mid n \in \omega \rangle$ is an element of L_β).

Let $B_n = \bigcup_{s \in A_{2n}^\star} B_{\sigma_s^{2n}}$ and $B = \bigcup_n B_n$. Since β is countable in L_γ , $\langle \sigma_s^n \mid n \in \omega \rangle$ is an element of L_β for each $s \in 2^{<\omega}$ and the sets B_σ have Borel codes in L_γ for all names $\sigma \in L_\beta$, uniformly in σ , the set B has a Borel code in L_γ .

Claim 3.14. $\mu(A \setminus B) = 0$.

Proof. We have $\frac{\mu(A \cap U_s)}{\mu(U_s)} > 1 - 2^{-2n}$ for all $s \in A_{2n}^\star$ by the definition of A_{2n} and $\frac{\mu(B_s \cap U_s)}{\mu(U_s)} > 1 - 2^{-2n}$ for all $s \in A_{2n}$ by the choice of σ_s^{2n} . Hence

$$\frac{\mu(A \cap B_s)}{\mu(A \cap U_s)} \geq \frac{\mu(A \cap B_s)}{\mu(U_s)} > 1 - 2^{-n}$$

for all $s \in A_{2n}$. Moreover,

$$\mu\left(\bigcup_{s \in A_n^*} (A \cap U_s)\right) = \mu\left(A \cap \bigcup_{s \in A_n^*} U_s\right) = \mu(A)$$

by Claim 3.13. Since $A_n^* \subseteq A_n$ is an antichain, the sets $A \cap U_s$ for $s \in A_n^*$ are pairwise disjoint. Therefore the previous inequality implies that

$$\frac{\mu(A \cap B_n)}{\mu(A)} > 1 - 2^{-n}.$$

Since $B = \bigcup_n B_n$, this implies $\frac{\mu(A \cap B)}{\mu(A)} = 1$ and hence $\mu(A \setminus B) = 0$. \square

Claim 3.15. $\varphi((\sigma_s^{2n})^x, \tau^x)$ holds in $L_\alpha[x]$.

Proof. We have $x \in A$ by the assumption. Since A and B have Borel codes in L_γ , $\mu(A \setminus B) = 0$ and x is random over L_γ , $x \in B$. Then $x \in B_n$ for some n and $x \in B_{\sigma_s^{2n}} = \llbracket \varphi(\sigma_s^{2n}, \tau) \rrbracket$ for some $s \in A_{2n}^*$. By Lemma 2.7, $\varphi((\sigma_s^{2n})^x, \tau)$ holds in $L_\alpha[x]$. \square

Hence the statement $\exists u \varphi(u, \tau^x)$ reflects to $L_\alpha[x]$. \square

The assumptions in Lemma 3.12 for $n = 2$ are not optimal for the application to ITTMs below. We will see in Section 4.1 that ITTM-randomness is a sufficient assumption for the applications.

3.3. Writable reals from non-null sets. We will prove an analogue to the following theorem for infinite time Turing machines. Let \leq_T denote Turing reducibility.

Theorem 3.16. (Sacks, see [DH10, Corollary 11.7.2]) If a real x is computable if and only if $\{y \mid x \leq_T y\}$ has positive Lebesgue measure.

In [CS17], analogues of this theorem for other machines were considered. It was asked if this holds for infinite time Turing machines, and this was only proved for non-meager Borel sets, via Cohen forcing over levels of the constructible hierarchy. With the results in Section 2, we prove this for Lebesgue measure.

Theorem 3.17. (1) A real x is writable if and only if $\mu(\{y : x \leq_w y\}) > 0$
 (2) A real x is eventually writable if and only if $\mu(\{y : x \leq_{ew} y\}) > 0$
 (3) A real x is accidentally writable if and only if $\mu(\{y : x \leq_{aw} y\}) > 0$

Proof. The forward direction is clear in each case. In the other direction, we only prove the writable case, since the proofs of the remaining cases are analogous.

Let $W_x := \{y : x \leq_w y\}$ and choose some sufficiently random $r \in W_x$. Since Σ is a limit of admissible ordinals (see [Wel09, Fact 2.5, Lemma 6]), $L_\Sigma[r] = L_\Sigma^r$ by Lemma 2.9 and Lemma 2.10 and $L_\Sigma[r]$ is an increasing union of admissible sets by Lemma 2.11. We choose some sufficiently random $s \in W_x$ over $L_\Sigma[r]$, in particular s is random over $L_{\Sigma+1}$. Since $L_\lambda \prec_{\Sigma_1} L_\zeta \prec_{\Sigma_2} L_\Sigma$, we have

$$L_\lambda[r] \prec_{\Sigma_1} L_\zeta[r] \prec_{\Sigma_2} L_\Sigma[r]$$

by Theorem 3.12, and we obtain the same elementary chain for s . Since $(\lambda^r, \zeta^r, \Sigma^r)$ and $(\lambda^s, \zeta^s, \Sigma^s)$ are lexically minimal and the values do not decrease in the extensions by r and s , this implies $\lambda = \lambda^r = \lambda^s$, $\zeta = \zeta^r = \zeta^s$ and $\Sigma = \Sigma^r = \Sigma^s$.

We can assume that r is random over L_γ and s is random over $L_\gamma[r]$ for some $\gamma > \Sigma$ such that L_γ satisfies a sufficiently strong theory to prove the forcing theorem and facts about random forcing, and such that generics and quasi-generics over L_γ coincide (see [Jec03a, Lemma 26.4]). Since the 2-step iteration of random forcing is equivalent to the

side-by-side random forcing (see [BJ95, Lemma 3.2.8]), (r, s) is side-by-side random over $L_{\Sigma+1}$.²

Since x is writable relative to r and relative to s , $x \in L_\lambda[r] \cap L_\lambda[s] = \lambda$ by Lemma 2.16, therefore x is writable. \square

As far as we know, the following class is the largest class between Π_1^1 and Σ_2^1 that has been studied. We write $x \leq_{n\text{-hyp}} y$ if x is computable from y by a Σ_n -hypermachine introduced in [?].

Theorem 3.18. For all $n \geq 1$, a real x is writable if and only if $\mu(\{y : x \leq_{n\text{-hyp}} y\}) > 0$

Proof. The proof is analogous to the proof of Theorem 3.17 via the results of [?] and the version of Lemma 3.12 for Σ_n -formulas instead of Σ_2 -formulas. \square

3.4. Recognizable reals from non-null sets. We will prove an analogous result as in the previous section, where computable reals are replaced with *recognizable reals* from [HL00]. This is an interesting and much stronger alternative notion to computability. The divergence between computability and recognizability is studied in [HL00, ?].

A real is recognizable if its singleton is decidable. *Lost melodies*, i.e. recognizable non-computable sets, do not appear in Turing computation, but already exists in the hyperarithmetic setting as Π_1^1 non-hyperarithmetic singletons.

Definition 3.19. (a) A real x is *recognizable* if and only if there is an ITTM-program P such that P halts for every input y , with output 1 if and only if $x = y$.
 (b) A real x is a *lost melody* if it is recognizable, but not writable.

A simple example for a lost melody is the constructibly least code for a model of $ZFC + V=L$. It was demonstrated in [Cara, Theorem 3.12] that every real that is recognizable from all elements of a non-meager Borel set is itself recognizable. The new observation for the following proof is that one can avoid computing generics by working with the forcing relation. This also leads to a simpler proof in the non-meager case.

Theorem 3.20. Suppose that a real x is recognizable from all elements of A and $\mu(A) > 0$. Then x is recognizable.

Proof. We can assume that there is a single program P which recognizes x from all oracles in A , since the set of oracles which recognize x for a fixed program is absolutely Δ_2^1 and hence Lebesgue measurable (see [Kan09, Exercise 14.4]).

Claim 3.21. Let D be the set of the conditions in L_{λ^x} which decide whether x is accepted or rejected by P relative to the random real over $L_{\Sigma^{x+1}}$. Then $\mu(A \setminus \bigcup D) = 0$.

Proof. If the conclusion fails, then there is a random real y over $L_{\Sigma^{x+1}}$ in $A \setminus \bigcup D$. Since $P^{x \oplus z}$ converges for any $z \in A$, $P^{x \oplus y} \downarrow = i$ for some i . Since $\lambda^{x \oplus y} = \lambda^x$ by Theorem 3.12 and $L_{\lambda^x}[x \oplus y] = L_{\lambda^x}^{x \oplus y}$ by Lemma 2.9 and Lemma 2.10, there is a name \dot{C} in L_{λ^x} and a condition p in L_{λ^x} with $y \in [p]$ which forces that \dot{C} is a computation of P with input $x \oplus y$ and output i . Then $p \in D$ and $y \in \bigcup D$, contradicting the assumption on y . \square

By the Lebesgue density theorem, there is an open interval with rational endpoints for which the relative measure of A is $> 1 - \epsilon$ for some $\epsilon < \frac{1}{3}$. We can assume that this interval is equal to ${}^\omega 2$.

The procedure Q for recognizing x works as follows. Suppose that \dot{y} is a name for the random real over $L_{\Sigma+1}$. Given an oracle z , we enumerate $L_{\lambda^z}[z]$ via a universal ITTM. In parallel, we search for pairs (p, \dot{C}) in $L_{\lambda^z}[z]$ such that p is a condition and \dot{C} is a name such that p forces over $L_{\lambda^z}[z]$ that \dot{C} is a computation of P in the oracle $z \oplus \dot{y}$. that halts

²Alternatively, the proof of the product lemma or the 2-step lemma [Jec03b, Lemma 15.9, Theorem 16.2] can easily be adapted to show directly that (r, s) is side-by-side random over $L_{\Sigma+1}$.

with output 0 or 1. Note that these are Δ_0 statements and that the forcing relation for such statements is Δ_1 by Lemma 2.6 and hence ITTM-decidable. We keep track of the conditions that force the corresponding computation to halt with output 0 or with output 1 on separate tapes. Moreover, we keep track of the measures u_0 and u_1 of the union of all conditions on the two tapes. Note that the measure of Borel sets can be computed in admissible sets by a Δ_1 -recursion and hence it is ITTM-computable. Since $\mu(A) > 1 - \epsilon$ and $\mu(A \setminus \bigcup D) = 0$, eventually $u_0 + u_1 > 1 - \epsilon$. As soon as this happens, we output 1 if $u_0 > 1 - 2\epsilon$ and 0 otherwise. We claim that Q^z outputs 1 if and only if $z = x$.

Claim 3.22. $Q^x \downarrow = 1$.

Proof. The measure of a countable union of sets can be approximated with arbitrary precision by unions of a finite number of sets. Since $\mu(A \setminus \bigcup D) = 0$ and $\mu(A) > 1 - \epsilon$, $\mu(\bigcup D) > 1 - \epsilon$. There are disjoint conditions $p, q \in L_{\lambda^x}[x]$ with $\mu([p] \cup [q]) > 1 - \epsilon$ such that p forces $Q^{x \oplus y} \downarrow = 1$, and q forces $P^{x \oplus y} \downarrow = 0$. Since $\mu(\bigcup D) > 1 - \epsilon$, $\mu([q]) \leq \epsilon$ and hence $\mu([p]) > 1 - 2\epsilon$. Eventually, such a condition p will be found and hence the procedure halts with output 1. \square

Claim 3.23. $Q^z \downarrow = 0$ if $z \neq x$.

Proof. Suppose that the claim fails. Since Q always halts, we have $Q^z \downarrow = 1$. Then there is a condition p with $\mu([p]) > 1 - 2\epsilon$ which forces $P^{z \oplus y} \downarrow = 1$. Since $\mu(A) > 1 - \epsilon$ and $\epsilon < \frac{1}{3}$, $\mu(A \cap [p]) > 0$ and hence there is a random y in $A \cap [p]$ over $L_{\lambda^z}[z]$. Since $y \in [p]$, $P^{z \oplus y} = 1$. Since $y \in A$ and $z \neq x$, $P^{z \oplus y} = 0$. \square

This completes the proof of Theorem 3.20. \square

The results in Section 2 also imply analogues of Theorem 3.17 and Theorem 3.20 for other notions of computation and recognizability, for instance the infinite time register machines [CFK⁺10] and a weaker variant [CFK⁺10]. We explore this in further work.

4. RANDOM REALS

We introduce natural randomness notions associated with infinite time Turing machines and show that they have various desirable properties.

This is the motivation for the previous results, which we will apply here. The results resemble the hyperarithmetic setting, although some proofs are different. Theorem 4.7 shows a difference to the hyperarithmetic case.

4.1. ITTM-random reals. The following is a natural analogue to Π_1^1 -random.

Definition 4.1. A real x is ITTM-random if it is not an element of any ITTM-semidecidable null set. The definition relativizes to reals.

We first note that there is a universal test. This follows from the following lemma, as in [HN07, Theorem 5.2].

Lemma 4.2. We can effectively assign to each ITTM-semidecidable set S an ITTM-semidecidable set \hat{S} with $\mu(\hat{S}) = 0$, and $\hat{S} = S$ if $\lambda(S) = 0$.

Proof. Suppose that S is an ITTM-semi-decidable set, given by a program P . We define S_α as the set of z such that $P(z)$ halts before α . Note that if M is admissible and contains a code for α , then there is a Borel code for S_α in M and hence $\mu(S_\alpha)$ can be calculated in M . In particular, $\mu(S_\alpha)$ is ITTM-writable from any code for α . Moreover, α is ITTM-writable in z since $\alpha < \lambda^z$. Hence there is a code for α in L_{λ^z} . Let \hat{S} be the set of all z such that there exists some $\alpha < \lambda^z$ with $z \in S_\alpha$ and $\mu(S_\alpha) = 0$. Moreover, let \hat{S}_α denote the set of z with $z \in S_\alpha$ and $\mu(S_\alpha) = 0$. Since the set of z with $\lambda^z = \lambda$ is co-null by Theorem 3.17, \hat{S} is the union of a null set and the sets \hat{S}_α for all $\alpha < \lambda$. \square

The universal test is the union of all sets \hat{S} , where S ranges over the ITTM-semidecidable sets. The following notion is analogous to Π_1^1 -random.

The following is a variant of van Lambalgen's theorem for ITTMs. We say that reals x and y are *mutually random*, in any given notion of randomness, if their join $x \oplus y$ is random.

Lemma 4.3. A real x is ITTM-random and a real y is ITTM-random relative to x if and only if x and y are mutually ITTM-random.

Proof. Suppose that x is ITTM-random and y is ITTM-random relative to x . Moreover, suppose that x and y are not mutual ITTM-randoms. Then there is an ITTM-semidecidable set A given by a program P such that $x \oplus y \in A$. Let $A_u = \{v \mid u \oplus v \in A\}$ denote the section of A at u . Let

$$A_{>q} := \{u \mid \mu(A_u) > q\}$$

for $q \in \mathbb{Q}$. Note that $u \in A_{>q}$ if and only if some condition in L_{Σ^u} with measure $r > q$ in \mathbb{Q} forces that $P(\check{u}, \check{v})$ halts, where \check{v} is a name for the random real over L_{Σ^u} , by Lemma 2.6. This is a Σ_1 -statement in L_{Σ^u} and therefore in L_{λ^u} . Then the set $A_{>q}$ is semidecidable by Lemma 3.4, uniformly in $q \in \mathbb{Q}$. Since $\mu(A) = 0$, $\mu(A_{>0}) = 0$. Since x is ITTM-random, $x \notin A_{>0}$ and hence $\mu(A_x) = 0$. Note that A_x is semidecidable in x . Since y is ITTM-random relative to x , this implies $y \notin A_x$, contradicting the assumption that $x \oplus y \in A$.

Now suppose that x and y are mutually ITTM-random. To show that x is ITTM-random, suppose that A is a semidecidable null set with $x \in A$. Then $A \oplus {}^\omega 2$ is a semidecidable null set containing $x \oplus y$, contradicting the assumption that x and y are mutually ITTM-random. To show that y is ITTM-random relative to x , suppose that y is an element of a semidecidable null set A relative to x . Since the construction of \hat{S} in Lemma 4.2 is effective, there is a semidecidable null subset B of ${}^\omega 2 \times {}^\omega 2$ with $A = B_x$ (in fact, all sections of B are null). Then $x \oplus y \in A$, contradicting the assumption that x and y are mutual ITTM-randoms. \square

The following result is analogous to the statement that a real x is Π_1^1 -random can be characterized by Δ_1^1 -randomness and $\omega_1^x = \omega_1^{\text{ck}}$ (see [Nie09, Theorem 9.3.9]).

Theorem 4.4. A real x is ITTM-random if and only if it is random over L_Σ and $\Sigma^x = \Sigma$. Moreover, this implies $\lambda^x = \lambda$.

Proof. First suppose that x is ITTM-random. We first claim that x is random over L_Σ . Since every real in L_Σ is accidentally writable, we can enumerate all Borel codes in L_Σ for sets A with $\mu(A) = 0$ and test whether x is an element of A . Therefore the set of reals which are not random over L_Σ is an ITTM-semidecidable set with measure 0, and hence x is random over L_Σ . We now claim that $\Sigma^x = \Sigma$. Since $\Sigma^y = \Sigma$ holds for all sufficiently random reals by Lemma 3.12, the set A of reals y with $\Sigma^y > \Sigma$ has measure 0. Since the existence of Σ is a Σ_1 -statement over L_{Σ^y} , the set A is semidecidable. Since x is ITTM-random, $x \notin A$ and hence $\Sigma^x = \Sigma$.

Second, suppose that x is random over L_Σ and $\Sigma^x = \Sigma$. Suppose that A is a semidecidable null set containing x given by a program P . Then $P(x)$ halts before $\lambda^x < \Sigma^x = \Sigma$ and hence some condition p forces over L_Σ that $P(x)$ halts, by Lemma 2.6. Then $\mu(A) > 0$, contradicting the assumption that A is null.

To show that $\lambda^x = \lambda$, note that $L_\lambda[x] \prec_{\Sigma_1} L_\Sigma[x] = L_{\Sigma^x}[x]$ by Lemma 3.12. Since λ^x is minimal with this property, $\lambda^x \leq \lambda$. \square

This shows that the level of randomness in the assumption of Lemma 3.12 can be improved to ITTM-random for $\alpha = \zeta$, $\beta = \Sigma$.

Surprisingly, we do not know if $\zeta^x = \zeta$ for ITTM-randoms x . This does not follow from the proof of Lemma 3.12, since the set \bar{A} defined in the beginning of the proof is not ITTM-semidecidable, but this would be needed for the proof of Claim 3.15 in the proof of Lemma 3.12.

We obtain the following variant of Theorem 3.17.

Theorem 4.5. If x is computable from both y and z and y is ITTM-random in z , then x is computable. In particular, this holds if y and z are mutual ITTM-randoms.

Proof. Suppose that $P(y) = Q(z) = x$. Then $A = \{u \mid P(u) = Q(z)\}$ is semidecidable in z . If $\mu(A) > 0$, then x is computable from all element of a set of positive measure and hence x is computable by Theorem 3.17. Suppose that $\mu(A) = 0$. Then $y \notin A$, since y is ITTM-random in z , contradicting the assumption that $y \in A$. \square

4.2. A decidable variant. Martin-Löf suggested to study Δ_1^1 -random reals. The following variant of ITTM-random is an analogue to Δ_1^1 -random.

Definition 4.6. A real is ITTM-decidable random if it is not an element of any decidable null set.

We now give a characterization of this notion. We call a real co-ITTM-random if it avoids the complement of every semidecidable set of measure 1. The following result is analogous to the equivalence of Δ_1^1 -random and Σ_1^1 -random [CY, Exercise 14.2.1].

Theorem 4.7. The following properties are equivalent.

- (a) x is co-ITTM-random.
- (b) x is ITTM-decidable random.
- (c) x is random over L_λ .

Proof. The first implication is clear.

For the second implication, note that since every Borel set with a Borel code in L_λ is ITTM-decidable, every ITTM-decidable random real x is random over L_λ .

For the remaining implication, suppose that x is random over L_λ and P is a program that decides the complement of a null set A with $x \in A$. Suppose that \dot{x} is the canonical name for the random real (note that this name is equal for randoms over arbitrary admissible sets). Relative to the set of random reals y over $L_{\Sigma+1}$, A is definable over L_Σ , since $\Sigma^y = \Sigma$ by Theorem 3.12. Hence $y \notin A$ and $P(y)$ halts before $\lambda^y = \lambda$ for any such real. Therefore in L_Σ , there is some γ (namely λ) such that the Boolean value of the statement that $P(\dot{x})$ halts strictly before γ is equal to 1. The existence of such an ordinal γ is a Σ_1 -statement, hence there is such an ordinal $\bar{\gamma} < \lambda$ such that the statement holds in L_λ for $\bar{\gamma}$, by Σ_1 -reflection. Let A denote the Boolean value of the statement that $P(\dot{x})$ halts before $\bar{\gamma}$. Then A is a Borel set with a Borel code in L_λ and $\mu(A) = 1$. Therefore $x \in A$ and $P(x)$ halts before λ , contradicting the assumption that $x \in A$. \square

Hence the distance between the analogues to Δ_1^1 -random and Π_1^1 -random is larger than for the original notions.

Lemma 4.8. There is no universal ITTM-decidable random test.

Proof. Suppose that A is a universal ITTM-decidable random test. In particular, the complement of A is ITTM-semidecidable. By the characterization of ITTM-semidecidable reals in Lemma 3.4 and [SS12, Seyfferth-Schlicht, Corollary 8], ITTM-semidecidable uniformization holds.³ Therefore, every semidecidable set, in particular the complement of A , has a recognizable element. This contradicts the assumption that A is a universal test. \square

³The proof is a variant of the proof of Π_1^1 -uniformization.

We call a program P *deciding* if $P(x)$ halts for every input x . The following is a version of von Lambalgen's theorem for ITTM-decidable.

Lemma 4.9. A real x is ITTM-decidable random and a real y is ITTM-decidable random relative to x if and only if $x \oplus y$ is ITTM-decidable random.

Proof. Suppose that $x \oplus y$ is ITTM-decidable random. The forward direction is a slight modification of the proof of von Lambalgen's theorem for ITTMs in Lemma 4.3, so we omit it. In the other direction, the only missing piece is the following claim.

Claim 4.10. Suppose that A is a decidable set given and $A_x = \{y \mid x \oplus y \in A\}$ is null. Then there is a decidable set B such that $A_x = B_x$ and all sections of B are null.

Proof. It was shown in the proof of Lemma 4.3 that the set

$$A_{>q} = \{u \mid \mu(A_u) > q\}$$

is semidecidable for all rationals q , uniformly in q , since the statement $u \in A_{>q}$ is Σ_1 over L_{Σ^u} . Since $L_{\lambda^u} \prec_{\Sigma_1} L_{\Sigma^u}$, the statement is Σ_1 over L_{λ^u} . Let

$$A_{\geq q} = \{u \mid \mu(A_u) \geq q\},$$

Then the statement $u \in A_{\geq q}$ is equivalent to $u \in A_{>r}$ for unboundedly many rationals $r < q$. Since λ^u is u -admissible, this is a Σ_1 -statement in u over L_{λ^u} . Hence $A_{\geq q}$ is semidecidable, uniformly in q .

Therefore, if A is decidable, then $A_{>q}$ and $A_{\geq q}$ are semidecidable, uniformly in q . Using the fact that $A_{=0} = \{u \mid \mu(A_u) = 0\}$ is decidable, it is easy to define a decidable set B as in the claim. \square

This completes the proof of Lemma 4.9. \square

Lemma 4.7 and 4.9 immediately imply that x and y are mutually random over L_λ if and only if x is random over L_λ and y is random over L_{λ^x} .

The following variant of Lemma 4.5 for reals computable from two mutually randoms can be shown for the following stronger reduction. A *safe ITTM-reduction* of a real x to a real y is a deciding ITTM (i.e. P halts on every input) with $P(x) = y$. We call reals x and y *mutually ITTM-decidable random* if $x \oplus y$ is ITTM-decidable random.

Lemma 4.11. If x is safely ITTM-reducible both to y and z , and y and z are mutually ITTM-decidable random, then x is ITTM-computable.

Proof. Suppose that P is a safe reduction of x to y and Q is a safe reduction of x to z . Since P is a safe reduction, the set $A = \{u \mid P(u) = Q(z)\}$ is ITTM-decidable relative to z . As $P(y) = x = Q(z)$, $y \in A$. Since y is ITTM-decidable relative to z , A is not null. Then P computes x from all elements of a non-null Lebesgue measurable set, and hence x is computable by 3.17. \square

Lemma 4.9 can be interpreted as the statement that x and y are mutually random (i.e. $x \oplus y$ is random) over L_λ if and only if x is random over L_λ and y is random over L_{λ^x} , by the relativized version of Lemma 4.7.

Intuitively, a random sequence should not be able to compute any non-computable sequence with special properties, such as recognizable sequences. The following result confirms this.

Lemma 4.12. Any recognizable real x that is computable from an ITTM-random real y is already computable.

Proof. Suppose that P recognizes x and $Q(y) = x$. Then the set

$$A = \{z \mid P^{Q(z)} = 1\}$$

is semi-decidable and contains y , where $Q(z)$ is the output of the computation Q with input z . Note that x is computable from every element of A via Q . If A is not null, then x is computable by Theorem 3.17. If A is null, this contradicts the assumption that y is ITTM-random and thus avoids A . \square

Hence there are real numbers that are not computable from any ITTM-random real, and therefore there is no analogue for ITTM-randoms to the Kučera-Gács theorem (see [DH10, Theorem 8.3.2]).

Remark 4.13. All previous results and proofs work relativized to reals and for arbitrary continuous measures instead of the Lebesgue measure.

4.3. Comparison with a Martin-Löf type variant. We finally consider a Martin-Löf variant of ITTM-randomness. The importance of this notion lies in its characterization via initial segment complexity. This variant is strictly between ITTM-random and Π_1^1 -random.

We first describe analogues of the theorems of van Lambalgen and Levin-Schnorr for ITTM_{ML}-random reals. Since these results are minor modifications of the results in [HN07] and [BGM], we refer the reader to [HN07, Section 3] and [BGM, Section 1.1, Section 3] for discussions and proofs, and will only point out the differences to our setting.

Towards proving van Lambalgen's theorem for ITTM_{ML}-random reals, we define a continuous relativization as in [BGM, Section 1.1]. If $\Psi \subseteq {}^\omega 2 \times {}^\omega 2$ and $x \in {}^\omega 2$, let

$$\Psi^{(x)} = \{n \mid (\sigma, m) \in \Psi \text{ for some } \sigma \preceq x\}.$$

A subset A of ${}^\omega 2$ is called ITTM_{ML}^(x) if $A = \Psi^{(x)}$ for some ITTM-semidecidable set Ψ .

Lemma 4.14. A real $x \oplus y$ is ITTM_{ML}-random if and only if x is ITTM_{ML}-random and y is ITTM_{ML}^(x)-random.

The difference to the proof in [BGM, Section 3] is that ω_1^{ck} is replaced with λ and the projectum function on ω_1^{ck} is replaced with a projectum function on λ , i.e. an injective function $p: \lambda \rightarrow \omega$ such that its graph is Σ_1 -definable over L_λ . For instance, consider the function p which maps an ordinal $\alpha < \lambda$ to the least program that writes a code for α .

The proof of the Levin-Schnorr theorem in [HN07, Theorem 3.9] easily adapts to our setting as follows, by replacing ω_1^{ck} with λ and Π_1^1 -random with ITTM_{ML}-random. We will also call an ITTM simply a *machine*.

Lemma 4.15. There is an effective list $\langle M_d \mid d \in \omega \setminus \{0\} \rangle$ of all prefix-free ITTMs.

Proof. We can effectively replace each machine P by a prefix-free machine \hat{P} , by simulating P on all inputs with increasing length. \square

Given such a list, we obtain a universal prefix-free machine U by defining $U(0^{d-1}1\sigma) = M_d(\sigma)$. We identify U with an semidecidable subset of $2^{<\omega} \times 2^{<\omega}$. The ITTM-version of Solomonoff-Kolmogorov complexity is defined as

$$K(x) = K_U(x) = \min\{|\sigma| \mid U(\sigma) = x\}.$$

Definition 4.16. Suppose that D is a prefix-free machine. The probability that D outputs a string x is $P_D(x) = \lambda(\{\sigma \mid D(\sigma) = x\})$.

By the definition of K , $2^{-K(x)} \leq P_U(x)$. As in [HN07, Theorem 3.4] we have the following result.

Theorem 4.17. (Coding theorem) For each prefix-free machine D , there is a Σ_1 -definable function $g: \lambda \rightarrow \lambda$ over L_λ and a constant c such that

$$\forall x \ 2^c 2^{-K(x)} \geq P_D(x).$$

This implies the following analogue to the Levin-Schnorr theorem to characterize randomness via incompressibility, as in [HN07, Theorem 3.9].

Theorem 4.18. The following properties are equivalent for infinite strings x .

- (a) x is ITTM_{ML} -random.
- (b) $\exists b \forall n K(x \upharpoonright n) > n - b$.

The difference to the proof of [HN07, Theorem 3.9] is that ω_1^{ck} is replaced with λ and the coding theorem for the ITTM-variant of K is used.

We now compare the introduced randomness notions with Π_1^1 -randomness. There is an ITTM-writable Π_1^1 -random real, for example, let x be the $<_L$ -least real that is random over $L_{\omega_1^{\text{ck}}+1}$. Since L_λ is admissible and ω_1^{ck} is countable in L_λ , $x \in L_\lambda$. Then x is Π_1^1 -random by Lemma 2.11 and Lemma 2.13, and all reals in L_λ are ITTM-computable.

For the next result, recall that a real $r \in \mathcal{R}$ is called *left- Π_1^1* if the set $\{q \in \mathbb{Q} \mid q \leq r\}$ is Π_1^1 . The following is a folklore result and we give a short proof for the benefit of the reader.

Lemma 4.19. (Tanaka, see [Kec73, Section 2.2 page 15]) The measure of Π_1^1 sets is uniformly left- Π_1^1 .

Proof. Using the Gandy-Spector theorem and Sacks' theorem (see Lemma 2.11) that the set of reals x with $\omega_1^x = \omega_1^{\text{ck}}$ has full measure, we can associate to a given Π_1^1 set a sequence of length ω_1^{ck} of hyperarithmetic subsets, such that their union approximates the set up to measure 0. This shows that the measure is left- Π_1^1 . Moreover, in the proof of the Gandy-Spector theorem (see [Hjo10, Theorem 5.5]) for a Π_1^1 set $\omega_2 \setminus p[T]$, the Σ_1 -formula states that T_x is well-founded, and hence the parameter in the formula is uniformly computable from T , and the assignment is uniform. \square

Lemma 4.20. Every ITTM-random is ITTM_{ML} -random and every ITTM_{ML} -random is Π_1^1 -random.

Proof. The first implication is obvious. For the second implication, suppose that $A = p[T]$ is a Σ_1^1 . Using Lemma 4.19, we inductively build finitely splitting subtrees S_n of T with $\mu([T] \setminus [S_n]) \leq 2^{-n}$, uniformly in n . This sequence can be written by an ITTM. \square

5. QUESTIONS

We conclude with several open questions. Surprisingly, the proof of Theorem 3.12 does not answer the following question.

Question 5.1. Is $\zeta^x = \zeta$ for every ITTM-random ζ ?

Moreover, we have left open various questions about the connections between randomness notions and their properties. The following question asks if a property of ML-random and Δ_1^1 -random (see [CY, Theorem 14.1.10]) holds in this setting.

Question 5.2. Is ITTM_{ML} -random strictly stronger than random over L_λ ?

The fact that ITTM_{ML} -random is strictly stronger than Π_1^1 -random suggests an analogue for Σ_n -hypermachines.

Question 5.3. Is every ML-random with respect to Σ_{n+1} -hypermachines already semidecidable random with respect to Σ_n -hypermachines?

Since the complexity of the set of Π_1^1 -randoms is Π_3^0 [Mon14, Corollary 27] and this is optimal (see [Mon14, Theorem 28] and [Yu11]), this suggests the following question.

Question 5.4. What is the complexity of the set of ITTM-random reals?

The set NCR is defined as the set of reals that are not random with respect to any continuous measure. It is known that this set has different properties in the hyperarithmetic setting [CY15] and for randomness over the constructible universe L [YZ].

Question 5.5. Is there concrete description of the set NCR, defined with respect to ITTM-randomness?

Moreover, it is open whether Theorem 4.5 fails for ITTM_{ML} -randomness. More precisely, we can ask for an analogue to the counterexample or ML-randomness (see [Nie09, Section 5.3]).

Question 5.6. Let Ω_0 and Ω_1 denote the halves of the ITTM-version of Chaitin's Ω (i.e. the halting probability for a universal prefix-free machine). Is some non-computable real computable from both Ω_0 and Ω_1 ?

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